GENERALIZED EULER CONSTANTS
FOR ARITHMETICAL PROGRESSIONS

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ABSTRACT. The work of Lehmer and Briggs on Euler constants in arithmetical progressions is extended to the generalized Euler constants that arise in the Laurent expansion of $\zeta(s)$ about $s = 1$. The results are applied to the summation of several classes of slowly converging series. A table of the constants is provided.

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ has the following Laurent expansion about its pole $s = 1$:

$$(1.1) \qquad \zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} A_k (s-1)^k.$$

The coefficients A_k are given by

$$A_k = \frac{(-1)^k}{k!} \gamma_k,$$

where

$$(1.2) \qquad \gamma_k = \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{\log^k j}{j} - \frac{\log^{k+1} n}{k+1} \right\}, \qquad k = 0, 1, 2, \dots$$

(we set $\log^0 1 = 1$). It is clear that $\gamma_0 = \gamma$, the Euler constant. The expansion (1.1) was independently established by several authors, and the γ_k have been studied quite extensively. For a short historical survey, see [3, p. 164 f.]; more recent references not included in [3] are [4, 11, 17, 18, 19]. The coefficients A_k are usually called Stieltjes constants, and the numbers γ_k are sometimes referred to as generalized Euler constants.

In this paper we shall consider generalizations of γ_k that arise by taking the sum in (1.2) over an arithmetical progression. More exactly, denote

$$(1.3) \qquad H_k(x, r, m) = \sum_{\substack{0 < n \le x \\ n \equiv r \ (\mathrm{mod}\ m)}} \frac{\log^k n}{n},$$

and define

$$(1.4) \qquad \gamma_k(r, m) := \lim_{x \to \infty} \left\{ H_k(x, r, m) - \frac{\log^{k+1} x}{m(k+1)} \right\}.$$

These constants were previously studied by Knopfmacher [14] as special cases of a wider class of generalized Euler constants. The case $k = 0$ was considered by Briggs [6] and Lehmer [15].

The purpose of this paper is to derive further properties of the constants $\gamma_k(r, m)$, several of them generalizing results in [6, 15]. This is done in §§2–4. In §5 (and already earlier) this will be applied to the summation of several classes of very slowly converging series; some of these results are known, while others appear to be new. In §6 we shall derive explicit expressions for $\gamma_1(r, m)$, $m = 3, 4$, and 6. A connection with a class of generalized gamma functions will be given in §7. Finally, in §8 we shall make some remarks on numerical calculations of the $\gamma_k(r, m)$; a table is provided, with $1 \le k \le 20$, $1 \le m \le 9$, and $1 \le r \le m$.

2. BASIC PROPERTIES OF THE $\gamma_k(r, m)$

The existence of the limit in (1.4) was deduced by Knopfmacher as an application of his more general theory in [14]. This and other results from [14] could also be proved directly by following the corresponding proofs for $k = 0$ in [15].

It follows immediately from (1.3) and (1.4) that

$$(2.1) \qquad \gamma_k(1, 1) = \gamma_k$$

and, since $H_k(x, r \pm m, m) = H_k(x, r, m)$, we have

$$(2.2) \qquad \gamma_k(r \pm m, m) = \gamma_k(r, m).$$

Because of (2.2), we may restrict our attention to $1 \le r \le m$.

Proposition 1 [14]. *Let $k \ge 0$, $m \ge 1$, and let d be a common divisor of r and m. Then*

$$(2.3) \qquad \gamma_k(r, m) = \frac{1}{d} \sum_{\nu=0}^{k} \binom{k}{\nu} (\log d)^{k-\nu} \gamma_\nu \left(\frac{r}{d}, \frac{m}{d} \right) - \frac{\log^{k+1} d}{m(k+1)}.$$

If we set $r = d = m$ and take (2.1) into account, we get the following special case of (2.3).

Corollary 1. *For all $k \ge 0$ and $m \ge 1$ we have*

$$(2.4) \qquad \gamma_k(m, m) = \frac{1}{m} \sum_{\nu=0}^{k} \binom{k}{\nu} (\log m)^{k-\nu} \gamma_\nu - \frac{\log^{k+1} m}{(k+1)m}.$$

In particular,

$$(2.5) \qquad \gamma_0(m, m) = \frac{1}{m} (\gamma - \log m),$$

$$(2.6) \qquad \gamma_1(m, m) = \frac{1}{m} \left(\gamma_1 + \gamma \log m - \frac{1}{2} \log^2 m \right).$$

Note that with $k = 0$ in (2.3) we get

$$\gamma_0(r, m) = \frac{1}{d}\gamma_0\left(\frac{r}{d}, \frac{m}{d}\right) - \frac{1}{m}\log d;$$

this is just Theorem 2 in [15]. Also, (2.5) is equation (2) in [15]. The next property is another analogue to a result (containing a misprint) in [15]; it is almost immediate from (1.4) and (1.3).

Proposition 2. *There holds*

$$(2.7) \qquad \sum_{j=0}^{n-1} \gamma_k(r + jm, nm) = \gamma_k(r, m).$$

In particular,

$$(2.8) \qquad \sum_{r=1}^{m} \gamma_k(r, m) = \gamma_k.$$

Proposition 3. *For $k \geq 0$ and $m \geq 1$ we have*

$$(2.9) \qquad \gamma_k(m, 2m) = \frac{1}{2m}\sum_{\nu=0}^{k}\binom{k}{\nu}\gamma_\nu(2\log^{k-\nu} m - \log^{k-\nu} 2m)$$
$$- \frac{2\log^{k+1} m - \log^{k+1} 2m}{2m(k+1)}.$$

In particular,

$$(2.10) \qquad \gamma_0(m, 2m) = \frac{1}{2m}\left(\gamma - \log\frac{m}{2}\right),$$

$$(2.11) \qquad \gamma_1(m, 2m) = \frac{1}{2m}\left(\gamma_1 + \gamma\log\frac{m}{2} + \frac{1}{2}\log^2 2m - \log^2 m\right).$$

Proof. Equation (2.7) with $n = 2$ and $r = 0$ gives

$$\gamma_k(m, 2m) = \gamma_k(0, m) - \gamma_k(0, 2m) = \gamma_k(m, m) - \gamma_k(2m, 2m).$$

The assertion (2.9) now follows from (2.4).   □

3. Series representations for $\gamma_k(r, m)$

We define the arithmetic function $g_r^{(m)}(n)$ by

$$g_r^{(m)}(n) = \begin{cases} m - 1 & \text{when } n \equiv r \pmod{m}, \\ -1 & \text{otherwise.} \end{cases}$$

Proposition 4. *For $k \geq 0$, $m \geq 2$, and $0 \leq r \leq m - 1$ we have*

$$(3.1) \qquad m\gamma_k(r, m) = \gamma_k + \sum_{n=1}^{\infty}\frac{\log^k n}{n}g_r^{(m)}(n).$$

Proof. Following [15], we denote for $0 \leq j \leq m - 1$,

$$(3.2) \qquad \sigma_j := \sum_{\lambda=0}^{m-1}\gamma_k(\lambda, m)e^{2\pi i\lambda j/m}.$$

Then by (2.8),

(3.3) $\sigma_0 = \gamma_k.$

To simplify notation, we write $\varepsilon := \exp(2\pi i/m)$. Let $j \neq 0$. We use the fact that

(3.4) $\displaystyle\sum_{\lambda=0}^{m-1} \varepsilon^{\lambda j} = 0.$

Then by (1.4) and (1.3) we have

$$\sigma_j = \lim_{x \to \infty} \left\{ \sum_{\lambda=0}^{m-1} \left(H_k(x, \lambda, m) - \frac{\log^{k+1} x}{(k+1)m} \right) \varepsilon^{\lambda j} \right\},$$

that is,

(3.5) $\displaystyle\sigma_j = \sum_{n=1}^{\infty} \frac{\log^k n}{n} \varepsilon^{nj}.$

We solve (3.2) for the $\gamma_k(\lambda, m)$ by multiplying both sides by ε^{-jr} and summing over j. Taking (3.4) into account, we get

$$\sum_{j=0}^{m-1} \sigma_j \varepsilon^{-jr} = \sum_{\lambda=0}^{m-1} \gamma_k(\lambda, m) \sum_{j=0}^{m-1} \varepsilon^{j(\lambda-r)} = m\gamma_k(r, m).$$

Hence, with (3.3) and (3.5) we get

(3.6) $\displaystyle m\gamma_k(r, m) = \gamma_k + \sum_{j=1}^{m-1} \varepsilon^{-jr} \sum_{n=1}^{\infty} \frac{\log^k n}{n} \varepsilon^{nj}.$

Now formally,

(3.7) $\displaystyle m\gamma_k(r, m) = \gamma_k + \sum_{n=1}^{\infty} \frac{\log^k n}{n} \sum_{j=1}^{m-1} \varepsilon^{j(n-r)}.$

The change in the order of summation of this conditionally convergent series can be justified by taking the appropriate limits for σ_j and the infinite series. Now by the definition of $g_r^{(m)}(n)$ and by (3.4) we have

$$\sum_{j=1}^{m-1} \varepsilon^{j(n-r)} = g_r^{(m)}(n).$$

The result (3.1) now follows from (3.7).   □

Remark. For an easier proof of (3.1), see the example after Proposition 9. However, we still require (3.6) and (3.7) elsewhere.

Proposition 5. *For* $k \geq 0$ *and* $m \geq 2$ *we have*

(3.8) $\displaystyle\sum_{n=1}^{\infty} \frac{\log^k n}{n} g_0^{(m)}(n) = \sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log m)^{k-\nu} \gamma_\nu - \frac{\log^{k+1} m}{k+1}.$

Proof. Take $r = 0$ in (3.1) and combine with (2.3).   □

We consider now the special cases $k = 1$ and (independently) $m = 2$. If we note that $g_0^{(2)}(n) = (-1)^n$, the following becomes obvious from (3.8).

Corollary 2. *For* $m \geq 2$ *we have*

$$(3.9) \qquad \gamma = \frac{1}{2} \log m + \frac{1}{\log m} \sum_{n=2}^{\infty} \frac{\log n}{n} g_0^{(m)}(n)$$

and in particular

$$(3.10) \qquad \gamma = \frac{1}{2} \log 2 + \frac{1}{\log 2} \sum_{n=2}^{\infty} (-1)^n \frac{\log n}{n}.$$

For $k \geq 0$ *we have*

$$(3.11) \qquad \sum_{n=1}^{\infty} (-1)^n \frac{\log^k n}{n} = \sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log 2)^{k-\nu} \gamma_\nu - \frac{\log^{k+1} 2}{k+1}.$$

Corollary 3. *If we set*

$$\tau_k^{(m)} := \sum_{n=1}^{\infty} \frac{\log^k n}{n} g_0^{(m)}(n), \qquad \tau_0(m) = -\log m,$$

then we have for all $m \geq 2$

$$(3.12) \qquad \gamma_k = \frac{1}{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} (\log m)^{j-1} B_j \tau_{k+1-j}^{(m)},$$

where B_j *is the* j*th Bernoulli number.*

Proof. This was shown in [16] for $m = 2$ by inverting the matrix of the triangular linear system (3.11). The proof carries over to arbitrary m without change, solving the linear system (3.8). □

Remarks. 1. While (3.10) is a well-known series expansion for the Euler constant (see, e.g., [10, p. 288]), (3.9) appears to be little known. It was previously proved by Kluyver [13] and rediscovered by Jacobsthal [12].

2. The expression (3.12) was proved by Kluyver [13]; note that a different notation for the Bernoulli numbers was used in [13].

3. The expansion (3.11) was proved by Liang and Todd [16]; it was used there to compute the $\tau_k = \tau_k^{(2)}$ for $1 \leq k \leq 20$ to 15 digits accuracy from the previously computed γ_k.

4. With $k = 0$ in (3.8) we get

$$\log m = -\sum_{n=1}^{\infty} \frac{1}{n} g_0^{(m)}(n)$$

$$(3.13) \qquad = 1 + \frac{1}{2} + \cdots + \frac{1}{m-1} - \frac{m-1}{m} + \frac{1}{m+1}$$

$$+ \cdots + \frac{1}{2m-1} - \frac{m-1}{2m} + \cdots;$$

this was proved in [15, p. 136; 12]; it is a generalization of the alternating harmonic series for $\log 2$.

Next we replace m by $2m$ in (3.1) and set $r = m$. Then we get the following result.

Proposition 6. *For* $k \geq 0$ *and* $m \geq 1$ *we have*

$$(3.14) \quad \sum_{n=1}^{\infty} \frac{\log^k n}{n} g_m^{(2m)}(n) = \sum_{\nu=0}^{k-1} \binom{k}{\nu} (2 \log^{k-\nu} m - \log^{k-\nu} 2m) \gamma_k$$
$$- \frac{1}{k+1} (2 \log^{k+1} m - \log^{k+1} 2m).$$

Again, we consider special cases, for $k = 0, 1, 2$ and for $m = 2$.

Corollary 4. *For* $m \geq 1$ *we have*

$$(3.15) \quad \sum_{n=1}^{\infty} \frac{1}{n} g_m^{(2m)}(n) = - \log \frac{m}{2},$$

$$(3.16) \quad \sum_{n=2}^{\infty} \frac{\log n}{n} g_m^{(2m)}(n) = \gamma \log \frac{m}{2} - \log^2 m + \frac{1}{2} \log^2 2m,$$

and in particular

$$(3.17) \quad \sum_{n=2}^{\infty} \frac{\log n}{n} g_2^{(4)}(n) = \log^2 2,$$

$$(3.18) \quad \sum_{n=2}^{\infty} \frac{\log^2 n}{n} g_m^{(2m)}(n) = \gamma(2 \log^2 m - \log^2 2m) + 2\gamma_1(2 \log m - \log 2m)$$
$$- \frac{2}{3} \log^3 m + \frac{1}{3} \log^3 2m,$$

and in particular

$$(3.19) \quad \gamma = \log 2 - \frac{1}{2 \log 2} \sum_{n=2}^{\infty} \frac{\log^2 n}{n} g_2^{(4)}(n).$$

Remarks. 1. With $m = 1$ in (3.14) we get (3.11) again; note that $g_1^{(2)}(n) = (-1)^{n-1}$.

2. Formula (3.15) is another generalization of the type (3.13) of the alternating harmonic series for $\log 2$. For $m = 2$ we get

$$-1 + \tfrac{3}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{3}{6} - \tfrac{1}{7} - \tfrac{1}{8} - \tfrac{1}{9} + \tfrac{3}{10} - \tfrac{1}{11} - \cdots = 0.$$

4. Primitive $\gamma_k(r, m)$

Following Lehmer [15], we call $\gamma_k(r, m)$ primitive when r and m are coprime. If $\gamma_k(r, m)$ is not primitive, then it can be expressed, via (2.3), in terms of primitive $\gamma_\nu(r_1, m_1)$, $0 \leq \nu \leq k$. This suggests a more detailed study of primitive $\gamma_k(r, m)$.

In agreement with the notation in [15], we set

$$(4.1) \quad \phi_k(m) := \sum_{\substack{(r, m)=1 \\ 1 \leq r \leq m}} \gamma_k(r, m).$$

For $j = 0, 1, 2, \ldots$ we introduce the sums

$$(4.2) \qquad N_j(m) := \sum_{d \mid m} \mu(d) \frac{m}{d} \log^j d,$$

where $\mu(d)$ is the Möbius function. In particular, we have

$$(4.3) \qquad N_0(m) = \sum_{d \mid m} \mu(d) = \varphi(m)$$

(see, e.g., [2, p. 26]) and

$$(4.4) \qquad N_1(m) = -\varphi(m) \sum_{p \mid m} \frac{\log p}{p - 1}$$

(see [15, p. 131]), where $\varphi(m)$ is the Euler totient function. The following result is a generalization of [15, Theorems 3–5].

Proposition 7. *For all $k \geq 0$ we have*

$$(4.5) \qquad m\phi_k(m) = \sum_{j=0}^{k} \binom{k}{j} \gamma_j N_{k-j}(m) - \frac{1}{k+1} N_{k+1}(m).$$

Proposition 7 could be proved by induction, following the proof of Theorem 3 in [15]. However it is easier to proceed as follows. We first need a lemma.

Lemma 1. *For all $k \geq 0$ and $m \geq 1$ we have*

$$(4.6) \qquad \sum_{(r,m)=1} \gamma_k(r, m) = \sum_{d \mid m} \mu(d)\gamma_k(0, d).$$

Proof. We use (2.7) with $r = 0$, $m = d$, and $n = m/d$, to get

$$\sum_{j=0}^{m/d-1} \gamma_k(jd, m) = \gamma_k(0, d) = \gamma_k(d, d).$$

Multiplying both sides by $\mu(d)$ and summing over all $d \mid m$, we obtain

$$\sum_{d \mid m} \mu(d)\gamma_k(d, d) = \sum_{d \mid m} \mu(d) \sum_{j=0}^{m/d-1} \gamma_k(jd, m)$$

$$= \sum_{r=0}^{m-1} \gamma_k(r, m) \sum_{d \mid (r,m)} \mu(d).$$

Here the inner sum was obtained by noting that d has to divide both r and m, hence the g.c.d. of r and m. By a basic property of the Möbius function (see, e.g., [2, p. 25]) the inner sum is zero unless $(r, m) = 1$. Thus we obtain the left-hand side of (4.6), and the proof is complete. $\square$

Example. Let $m = 6$ and $k = 0$. Then (4.6) becomes

$$\gamma(1, 6) + \gamma(5, 6) = \gamma(1, 1) - \gamma(2, 2) - \gamma(3, 3) + \gamma(6, 6).$$

This agrees with Table 1 in [15].

Proof of Proposition 7. In (2.4) we replace m by d, multiply both sides by $\mu(d)$, and sum over all $d \mid m$. Then, by (4.1) and (4.6), we have

$$m\phi_k(m) = \sum_{d \mid m} \mu(d) \frac{m}{d} \left\{ \sum_{\nu=0}^{k} \binom{k}{\nu} (\log d)^{k-\nu} \gamma_\nu - \frac{\log^{k+1} d}{k+1} \right\}.$$

If we now change the order of summation and use (4.2), we get the right-hand side of (4.5). This completes the proof. □

Remarks. 1. With $k = 0$ in (4.5), and using (4.3) and (4.4), we get

$$m\phi_0(m) = \varphi(m) \left\{ \gamma + \sum_{p \mid m} \frac{\log p}{p-1} \right\}.$$

This is Theorem 3 combined with Theorem 5 in [15]; see also [6].

2. With $m = 1$, (4.5) reduces to (2.1).

3. Knopfmacher [14] proved the following version of (4.5). For prime numbers p, define

$$c_0(p) := 1 - \frac{1}{p} \quad \text{and} \quad c_i(p) := (-1)^{i+1} \frac{\log^i p}{p} \quad (i \geq 1).$$

Now for integers $m \geq 1$ denote

$$a_j^{(m)} := \sum \prod_{p \mid m} c_{i_p}(p),$$

where the sum is taken over all i_p with $\sum_p i_p = j$. Then, in our notation,

$$(4.7) \qquad \frac{(-1)^k}{k!} \phi_k(m) = a_{k+1}^{(m)} + \sum_{j=0}^{k} \frac{k!}{j!} (-1)^{k-j} \gamma_j a_{k-j}^{(m)}.$$

Incidentally, comparing (4.5) and (4.7), we can prove by induction (beginning with $a_0^{(m)} = N_0(m)/m$) that

$$N_j(m) = (-1)^j m j! a_j^{(m)}.$$

Lehmer showed in [15] that if δ is a divisor of m, then $\phi_0(\delta m) = \phi_0(m)$. This property holds also for $\phi_k(m)$, $k \geq 1$.

Corollary 5. *Let* $k \geq 0$, $m \geq 1$, *and* $\delta \mid m$. *Then* $\phi_k(\delta m) = \phi_k(m)$.

Proof. Any divisor d of δm that is not already a divisor of m has to be divisible by a square. But $\mu(d) = 0$ if d is not square free. Hence, by (4.2) we have $N_j(\delta m) = \delta N_j(m)$ for $j = 0, 1, 2, \ldots$. The result follows from (4.5). □

The next application of Proposition 7 is an analogue of Theorem 6 in [15].

Corollary 6. *For* $k \geq 0$ *and* $m \geq 2$ *we have*

$$(4.8) \qquad \begin{aligned} &\sum_{j=1}^{m-1} \left(\sum_{n=1}^{\infty} \frac{\log^k n}{n} \cos \frac{2\pi nj}{m} \right) \varphi(m) \frac{\mu(m/(m,j))}{\varphi(m/(m,j))} \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \gamma_j N_{k-j}(m) - \frac{N_{k+1}(m)}{k+1}. \end{aligned}$$

Proof. We sum (3.6) over all $r < m$ relatively prime to m. Then, with (4.1), we get

$$(4.9) \qquad m\phi_k(m) = \varphi(m)\gamma_k + \sum_{j=1}^{m-1} \left(\sum_{n=1}^{\infty} \frac{\log^k n}{n} \varepsilon^{nj} \right) \sum_{(r,m)=1} e^{-2\pi \imath jr/m}.$$

The last sum in (4.9) is a Ramanujan sum $c_m(j)$, evaluated by Hölder's formula

$$c_m(j) = \varphi(m) \frac{\mu(m/(m,j))}{\varphi(m/(m,j))}$$

(see, e.g., [2, p. 164]). Since all the other terms in (4.9) are real, we may replace the infinite sum by its real part. We get now (4.8) by applying (4.5). □

Remark. If in (4.8) we let m be a prime p or a power of a prime, we will get (3.8) for $m = p$. To see this, we use the fact that

$$\sum_{j=0}^{m-1} \cos \frac{2\pi rj}{m} = 0 \quad \text{if } r \equiv 0 \pmod{m},$$

and apply the definition of $g_r^{(m)}(n)$. Also note that with (4.2) we have for integers α,

$$N_j(p^\alpha) = p^{\alpha-1} \log^j p, \qquad j = 0, 1, 2, \ldots.$$

If m has at least two different prime factors, (4.8) will give new summation formulas. However, it will be more convenient to use (3.1) for this purpose.

Corollary 7. *For $k \geq 0$ and $m \geq 2$ we have*

$$(4.10) \qquad \sum_{n=1}^{\infty} \frac{\log^k n}{n} \sum_{(m,r)=1} g_r^{(m)}(n) = \sum_{j=0}^{k-1} \binom{k}{j} \gamma_j N_{k-j}(m) - \frac{1}{k+1} N_{k+1}(m),$$

and in particular

$$(4.11) \qquad \sum_{n=1}^{\infty} \frac{1}{n} \sum_{(m,r)=1} g_r^{(m)}(n) = \varphi(m) \sum_{p \mid m} \frac{\log p}{p-1},$$

$$(4.12) \qquad \begin{aligned} \sum_{n=1}^{\infty} \frac{\log n}{n} &\sum_{(m,r)=1} g_r^{(m)}(n) \\ &= -\gamma\varphi(m) \sum_{p \mid m} \frac{\log p}{p-1} - \frac{1}{2} \sum_{d \mid m} \mu(d) \frac{m}{d} \log^2 d. \end{aligned}$$

Proof. Sum both sides of (3.1) over all $r < m$ relatively prime to m, and apply (4.1) and (4.5). The change in the order of summation can be justified by going back to the limits defining the $\gamma_k(r,m)$. □

Example. Let $m = 6$, the smallest positive integer with two different prime divisors. Let

$$h^{(6)}(n) = \sum_{(6,r)=1} g_r^{(6)}(m) = g_1^{(6)}(n) + g_5^{(6)}(n) = \begin{cases} 4 & \text{if } n \equiv 1, 5 \pmod 6, \\ -2 & \text{otherwise.} \end{cases}$$

Then, with (4.11) and (4.12), we get respectively

$$\sum_{n=1}^{\infty} \frac{1}{n} h^{(6)}(n) = 2\log 2 + \log 3,$$

$$\sum_{n=1}^{\infty} \frac{\log n}{n} h^{(6)}(n) = (\log 3 - 2\log 2)\gamma + \frac{1}{2}(3\log^2 2 + 2\log^2 3 - \log^2 6).$$

We note that if $h^{(m)}(n)$ is defined as above, we always have

$$h^{(m)}(n) = \begin{cases} m - \varphi(m) & \text{if } (n, m) = 1, \\ -\varphi(m) & \text{if } (n, m) > 1. \end{cases}$$

5. Summation of certain series

In this section we consider two applications of the $\gamma_k(r, m)$ to the summation of series involving kth powers of the logarithm. The first application is an analogue to Theorem 9 in [15].

Proposition 8. *Let* $k \geq 0$, $N \geq 2$, *and let* $(r_1, m_1), (r_2, m_2), \ldots, (r_N, m_N)$ *be pairs of positive integers for which* $1 \leq r_j \leq m_j$, $j = 1, 2, \ldots, N$, *and such that the rational numbers* r_j/m_j *are distinct. Then the series*

$$(5.1) \qquad S_k := \sum_{n=0}^{\infty} \left\{ c_1 \frac{\log^k(m_1 n + r_1)}{m_1 n + r_1} + \cdots + c_N \frac{\log^k(m_N n + r_N)}{m_N n + r_N} \right\}$$

converges if and only if

$$(5.2) \qquad \sum_{j=1}^{N} \frac{c_j}{m_j} \log^\nu m_j = 0 \quad \text{for } \nu = 0, 1, \ldots, k.$$

If the series (5.1) *converges, then*

$$(5.3) \qquad S_k = \sum_{j=1}^{N} c_j \left\{ \gamma_k(r_j, m_j) + \frac{1}{m_j} \frac{\log^{k+1} m_j}{k + 1} \right\}.$$

Proof. We adapt the proof of Theorem 9 in [15]. We have

$$S_k = \lim_{x \to \infty} \sum_{j=1}^{N} c_j \sum_{0 \leq n \leq x} \frac{\log^k(m_j n + r_j)}{m_j n + r_j}$$

$$= \lim_{x \to \infty} \sum_{j=1}^{N} c_j H_k(m_j x, r_j, m_j)$$

$$= \sum_{j=1}^{N} c_j \lim_{x \to \infty} \left\{ H_k(m_j x, r_j, m_j) - \frac{\log^{k+1}(m_j x)}{(k+1)m_j} \right\}$$

$$+ \lim_{x \to \infty} \sum_{j=1}^{N} \frac{c_j}{m_j} \frac{\log^{k+1}(m_j x)}{k + 1}.$$

Now we use (1.4), and in the second sum we expand $\log^{k+1}(m_j x)$ with the binomial theorem. Then, after a change in the order of summation, we get

$$S_k = \sum_{j=1}^{N} c_j \gamma_k(r_j, m_j) + \sum_{j=1}^{N} \frac{c_j}{m_j} \frac{\log^{k+1} m_j}{k+1}$$

$$+ \lim_{x \to \infty} \left\{ \sum_{\nu=0}^{k} \frac{1}{k+1} \binom{k+1}{\nu} \left(\sum_{j=1}^{N} \frac{c_j}{m_j} \log^{\nu} m_j \right) \log^{k+1-\nu} x \right\}.$$

It is now clear that the limit exists if and only if (5.2) holds for $\nu = 0, 1, \ldots, k$. This proves Proposition 8. $\square$

Example. Let

$$S_1 := \sum_{n=0}^{\infty} \left\{ \frac{\log(n+1)}{n+1} - 2\frac{\log(2n+1)}{2n+1} + \frac{\log 2}{2n+1} \right\}.$$

We can rewrite

$$S_1 = \sum_{n=0}^{\infty} \left\{ \frac{\log(n+1)}{n+1} - 4\frac{\log(2n+1)}{2n+1} + 4\frac{\log(4n+2)}{4n+2} \right\}$$

and verify that (5.2) holds. Hence the series converges, and we have by (5.3),

$$S_1 = \gamma_1(1, 1) - 4\left\{ \gamma_1(1, 2) + \frac{1}{2}\frac{\log^2 2}{2} \right\} + 4\left\{ \gamma_1(2, 4) + \frac{1}{4}\frac{\log^2 4}{2} \right\}.$$

Finally, with (2.1) and (2.11), we get

$$S_1 = (3\log 2 + 2\gamma)\log 2.$$

We consider now the special case where all the m_j are equal. Then (5.2) reduces to the condition that the sum of the c_j be zero, and consequently the logarithmic terms in (5.3) disappear. Hence we have

Corollary 8. *Let* $k \geq 0$, $N \geq 2$, *and let* m *and* $r_1, r_2, \ldots, r_N$ *be positive integers such that* $0 < r_1 < r_2 < \cdots < r_N \leq m$. *Let* $c_1, c_2, \ldots, c_N$ *be real or complex numbers such that* $c_1 + c_2 + \cdots + c_N = 0$. *Then*

$$(5.4) \qquad \sum_{n=0}^{\infty} \left\{ c_1 \frac{\log^k(mn+r_1)}{mn+r_1} + \cdots + c_N \frac{\log^k(mn+r_N)}{mn+r_N} \right\} = \sum_{j=1}^{N} c_j \gamma_k(r_j, m).$$

In two particular cases, the sum of the series can be given in terms of the γ_j, $j = 0, 1, \ldots, k$.

Corollary 9. *For* $k \geq 0$ *and* $m \geq 2$ *we have*

$$(5.5) \qquad \sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{m-1} \frac{\log^k(mn+j)}{mn+j} - (m-1)\frac{\log^k(m(n+1))}{m(n+1)} \right\}$$

$$= -\sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log m)^{k-\nu} \gamma_\nu + \frac{\log^{k+1} m}{k+1}.$$

Proof. We use (5.4) with $c_1 = c_2 = \cdots = c_{m-1} = 1$, $c_m = 1 - m$. With (2.8) and (2.4), the right-hand side of (5.4) becomes

$$\sum_{j=1}^{m-1} \gamma_k(j, m) - (m-1)\gamma_k(m, m) = \sum_{j=1}^{m} \gamma_k(j, m) - m\gamma_k(m, m)$$

$$= \gamma_k - \sum_{\nu=0}^{k} \binom{k}{\nu} (\log m)^{k-\nu}\gamma_\nu + \frac{1}{k+1} \log^{k+1} m.$$

This proves (5.5), since the terms involving γ_k are cancelled. $\square$

Example. For $k = 1$, the right-hand side of (5.5) is simply $((\log m)/2 - \gamma) \log m$.

Corollary 10. *For $k \geq 0$ and even integers $m \geq 4$ we have*

$$\sum_{n=0}^{\infty} \left\{ \sum_{(j, m)=1} \frac{\log^k(mn+j)}{mn+j} - \varphi(m)\frac{\log^k(mn+m/2)}{mn+m/2} \right\}$$

(5.6)
$$= \frac{1}{m} \sum_{j=0}^{k} \binom{k}{j} \gamma_j \left\{ N_{k-j}(m) - \varphi(m) \left(2\log^{k-j}\frac{m}{2} - \log^{k-j} m \right) \right\}$$

$$- \frac{1}{m(k+1)} \left\{ N_{k+1}(m) - \varphi(m) \left(2\log^{k+1}\frac{m}{2} - \log^{k+1} m \right) \right\},$$

where the $N_j(m)$ are defined by (4.2). In particular,

$$\sum_{n=0}^{\infty} \left\{ \sum_{(j, m)=1} \frac{\log(mn+j)}{mn+j} - \varphi(m)\frac{\log(mn+m/2)}{mn+m/2} \right\}$$

(5.7)
$$= -\gamma\frac{\varphi(m)}{m} \left\{ \sum_{p \,|\, m} \frac{\log p}{p-1} + \log m - 2\log 2 \right\}$$

$$- \frac{1}{2m} \left\{ \sum_{d \,|\, m} \mu(d)\frac{m}{d} \log^2 d - \varphi(m) \left(2\log^2\left(\frac{m}{2}\right) - \log^2 m \right) \right\}.$$

Proof. We apply (5.4) with $N = m$, $r_j = j$ for $j = 1, 2, \ldots, m$, $c_j = 1$ when $(j, m) = 1$, $c_j = 0$ otherwise except $c_{m/2} = -\varphi(m)$. Then the right-hand side of (5.4) becomes $\phi_k(m) - \varphi(m)\gamma_k(m/2, m)$, where $\phi_k(m)$ is defined by (4.1). We obtain now the right-hand side of (5.6) from (4.5) and (2.9). With (4.4), we immediately get (5.7). $\square$

Example. With $m = 6$, the relation (5.7) becomes after some calculation

$$\sum_{n=0}^{\infty} \left\{ \frac{\log(6n+1)}{6n+1} - 2\frac{\log(6n+3)}{6n+3} + \frac{\log(6n+5)}{6n+5} \right\} = \frac{\log 3}{2} \left(\frac{1}{2} - \log 2 - \gamma \right).$$

The second application of the $\gamma_k(r, m)$ is an analogue of Theorem 8 in [15], proved by Knopfmacher [14].

Proposition 9 [14]. *Let $g(n)$ be an arithmetical function, periodic with period m. Then*

$$(5.8) \qquad S_k(g) := \sum_{n=1}^{\infty} g(n) \frac{\log^k n}{n}$$

converges if and only if

$$(5.9) \qquad \sum_{j=1}^{m} g(j) = 0.$$

If $S_k(g)$ converges, then

$$(5.10) \qquad S_k(g) = \sum_{r=1}^{m} g(r)\gamma_k(r, m).$$

Example. The function $g(n) = g_r^{(m)}(n)$, as defined in §3, satisfies (5.9). Hence, Proposition 9 implies (3.1) upon applying (2.8).

A second application of Proposition 9 is given in the next section.

6. SOME EXPLICIT EXPRESSIONS

Let $g(n) = \chi(n)$ be a nonprincipal Dirichlet character modulo m, and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

the corresponding Dirichlet L-series. Then its kth derivative is

$$L^{(k)}(s, \chi) = (-1)^k \sum_{n=1}^{\infty} \chi(n) \frac{\log^k n}{n^s},$$

and with Proposition 9 we get the following, as was noted by Knopfmacher [14].

Proposition 10 [14]. *For $k \geq 0$ we have*

$$(6.1) \qquad L^{(k)}(1, \chi) = (-1)^k \sum_{r=1}^{m} \chi(r)\gamma_k(r, m).$$

For $k = 0$, this was shown by Lehmer [15]. The connection between $L(1, \chi)$ and the class number of quadratic fields enabled Lehmer in [15] to give explicit expressions for the right-hand side of (6.1) in the case $k = 0$.

If $k = 1$ and χ is an odd character (i.e., $\chi(-1) = -\chi(1)$) modulo f, then $L'(1, \chi)$ can be expressed in terms of classical functions (see, e.g., [8, p. 182]),

$$(6.2) \qquad L'(1, \chi) = i\pi \frac{\tau(\chi)}{f} \left\{ (\gamma + \log 2\pi)B_{1,\bar{\chi}} + \sum_{\nu=1}^{f} \bar{\chi}(\nu) \log \Gamma\left(\frac{\nu}{f}\right) \right\},$$

where

$$(6.3) \qquad \tau(\chi) = \sum_{\nu=0}^{f} \chi(\nu)e^{2\pi i\nu/f}$$

is the normalized Gauss sum associated with χ, and $B_{1,\chi}$ is the first generalized Bernoulli number belonging to the character χ; it can be expressed as

$$(6.4) \qquad B_{1,\chi} = \frac{1}{f} \sum_{\nu=1}^{f} \chi(\nu)\nu.$$

In the special case where χ is an odd quadratic character, we have $\bar{\chi} = \chi$ and $\tau(\chi) = i\sqrt{f}$ (see, e.g., [5, p. 349]). Also, since $\chi(f-\nu) = -\chi(\nu)$, and using the well-known functional equation $\Gamma(1-z)\Gamma(z) = \pi/\sin\pi z$, we have

$$\sum_{\nu=1}^{f} \chi(\nu)\log\Gamma\left(\frac{\nu}{f}\right) = \sum_{\nu=1}^{[f/2]} \chi(\nu)\log\frac{\Gamma(\nu/f)}{\Gamma((f-\nu)/f)}$$

$$= \sum_{\nu=1}^{[f/2]} \chi(\nu)\log\left(\Gamma\left(\frac{\nu}{f}\right)^2 \frac{\sin(\pi\nu/f)}{\pi}\right).$$

Hence, with (6.2), we get

$$(6.5) \qquad L'(1,\chi) = -\frac{\pi}{\sqrt{f}}\left\{(\gamma + \log 2\pi)B_{1,\chi} + \sum_{\nu=1}^{[f/2]} \chi(\nu)\log\left[\Gamma\left(\frac{\nu}{f}\right)^2 \frac{\sin(\pi\nu/f)}{\pi}\right]\right\}.$$

Example 1. Let $f = 3$, i.e., $\chi(1) = 1$, $\chi(2) = -1$, $\chi(3) = 0$. Then from (6.4) we have $B_{1,\chi} = -\frac{1}{3}$, and we get with (6.5)

$$L'(1,\chi) = -\frac{\pi}{\sqrt{3}}\left\{(\gamma + \log 2\pi)\frac{-1}{3} + \log\left[\Gamma\left(\frac{1}{3}\right)^2 \frac{\sqrt{3}}{2\pi}\right]\right\}.$$

Now by (6.1) we have

$$(6.6) \qquad \gamma_1(1,3) - \gamma_1(2,3) = -L'(1,\chi).$$

On the other hand, (2.6) gives

$$\gamma_1(0,3) = \gamma_1(3,3) = \frac{1}{3}\left(\gamma_1 + \gamma\log 3 - \frac{1}{2}\log^2 3\right),$$

and with (2.8) we have

$$(6.7) \qquad \gamma_1(1,3) + \gamma_1(2,3) = \gamma_1 - \gamma_1(0,3) = \frac{2}{3}\gamma_1 - \frac{1}{3}\gamma\log 3 + \frac{1}{6}\log^2 3.$$

Finally, we combine (6.6) and (6.7) to get

$$(6.8) \qquad \gamma_1(1,3) = \frac{1}{2}\left\{\frac{2}{3}\gamma_1 - \frac{1}{3}\gamma\log 3 + \frac{1}{6}\log^2 3 - L'(1,\chi)\right\},$$

$$(6.9) \qquad \gamma_1(2,3) = \frac{1}{2}\left\{\frac{2}{3}\gamma_1 - \frac{1}{3}\gamma\log 3 + \frac{1}{6}\log^2 3 + L'(1,\chi)\right\}.$$

Example 2. Let $f = 4$, i.e., $\chi(1) = 1$, $\chi(3) = -1$, $\chi(2) = \chi(4) = 0$. Then (6.4) gives $B_{1,\chi} = -\frac{1}{2}$, and (6.5) gives

$$L'(1,\chi) = \frac{-\pi}{2}\left\{(\gamma + \log 2\pi)\frac{-1}{2} + \log\left[\Gamma\left(\frac{1}{4}\right)^2 \frac{\sqrt{2}}{2\pi}\right]\right\}.$$

Now by (6.1) we have

(6.10)
$$\gamma_1(1, 4) - \gamma_1(3, 4) = -L'(1, \chi).$$

From (2.6) and (2.11) we get

$$\gamma_1(2, 4) = \tfrac{1}{4}(\gamma_1 + \log^2 2),$$
$$\gamma_1(4, 4) = \tfrac{1}{4}(\gamma_1 + 2\gamma \log 2 - 2\log^2 2),$$

and (2.8) gives

(6.11)
$$\gamma_1(1, 4) + \gamma_1(3, 4) = \gamma_1 - \gamma_1(2, 4) - \gamma_1(4, 4)$$
$$= \tfrac{1}{2}\gamma_1 - \tfrac{1}{2}\gamma \log 2 + \tfrac{1}{4}\log^2 2.$$

We combine (6.10) and (6.11) to get

$$\gamma_1(1, 4) = \tfrac{1}{2}\{\tfrac{1}{2}\gamma_1 - \tfrac{1}{2}\gamma \log 2 + \tfrac{1}{4}\log^2 2 - L'(1, \chi)\},$$
$$\gamma_1(3, 4) = \tfrac{1}{2}\{\tfrac{1}{2}\gamma_1 - \tfrac{1}{2}\gamma \log 2 + \tfrac{1}{4}\log^2 2 + L'(1, \chi)\}.$$

Example 3. Let $f = 6$, i.e., $\chi(1) = 1$, $\chi(5) = -1$, $\chi(n) = 0$ for all other n (mod 6). Note that the character χ is not primitive; we can write

$$L(s, \chi) = (1 + 2^{-s})L(s, \psi),$$

where ψ is the quadratic character of conductor 3 from Example 1. Differentiating both sides, we get

$$L'(1, \chi) = \frac{3}{2}L'(1, \psi) - \frac{\log 2}{2}L(1, \psi).$$

To determine $L(1, \psi)$, we use the well-known generalized Euler formula

$$L(1, \psi) = i\pi\frac{\tau(\psi)}{f}B_{1, \psi}$$

(see, e.g., [8, p. 182]). With $\tau(\psi) = i\sqrt{f}$, $f = 3$, and $B_{1, \psi} = -\tfrac{1}{3}$, this becomes $L(1, \psi) = \sqrt{3}\pi/9$, so that

(6.12)
$$L'(1, \chi) = \frac{3}{2}L'(1, \psi) - \frac{\log 2}{18}\sqrt{3}\pi.$$

With (2.3), we have now for $r = 1$ and $r = 2$,

$$\gamma_1(2r, 6) = \frac{\log 2}{2}\gamma_0(r, 3) + \frac{1}{2}\gamma_1(r, 3) - \frac{1}{12}\log^2 2.$$

In [15] we find, with $\varepsilon(1) = 1$, $\varepsilon(2) = -1$,

$$\gamma_0(r, 3) = \frac{1}{3}\gamma + \varepsilon(r)\frac{\pi}{18}\sqrt{3} + \frac{1}{6}\log 3,$$

and with (6.8) and (6.9) we get

(6.13)
$$\gamma_1(2, 6) = \frac{\log 2}{6}\left(\gamma + \frac{\pi}{6}\sqrt{3} + \frac{\log 3}{2}\right)$$
$$+ \frac{1}{12}\left(2\gamma_1 - \gamma \log 3 + \frac{1}{2}\log^2 3 - \log^2 2\right) - \frac{1}{4}L'(1, \psi),$$

$$\gamma_1(4,6) = \frac{\log 2}{6}\left(\gamma - \frac{\pi}{6}\sqrt{3} + \frac{1}{2}\log 3\right)$$

(6.14)

$$+ \frac{1}{12}\left(2\gamma_1 - \gamma\log 3 + \frac{1}{2}\log^2 3 - \log^2 2\right) + \frac{1}{4}L'(1,\psi).$$

Also, by (2.11) and (2.6) we have

(6.15) $\gamma_1(3,6) = \frac{1}{6}\left(\gamma_1 + \gamma\log\frac{3}{2} + \frac{1}{2}\log^2 6 - \log^2 3\right),$

(6.16) $\gamma_1(6,6) = \frac{1}{6}\left(\gamma_1 + \gamma\log 6 - \frac{1}{2}\log^2 6\right).$

Now, with (2.8) and (6.13)–(6.16), we get

$$\gamma_1(1,6) + \gamma_1(5,6) = \gamma_1 - [\gamma_1(2,6) + \gamma_1(4,6)] - [\gamma_1(3,6) + \gamma_1(6,6)]$$

(6.17)

$$= \frac{1}{3}\{\gamma_1 - \gamma(\log 2 + \frac{1}{2}\log 3) - \frac{1}{2}\log 2\log 3$$

$$+ \frac{1}{2}\log^2 2 + \frac{1}{4}\log^2 3\}.$$

On the other hand, by (6.1) and (6.12), we have

$$\gamma_1(1,6) - \gamma_1(5,6) = -\frac{3}{2}L'(1,\psi) + \frac{\log 2}{18}\sqrt{3}\pi,$$

and combining this with (6.17), we obtain

$$\gamma_1(1,6) = \frac{1}{6}\left\{\gamma_1 - \gamma\left(\log 2 + \frac{1}{2}\log 3\right) - \frac{1}{2}\log 2\log 3 + \frac{1}{2}\log^2 2 + \frac{1}{4}\log^2 3\right\}$$

$$- \frac{3}{4}L'(1,\psi) + \frac{\log 2}{36}\sqrt{3}\pi,$$

$$\gamma_1(5,6) = \frac{1}{6}\left\{\gamma_1 - \gamma\left(\log 2 + \frac{1}{2}\log 3\right) - \frac{1}{2}\log 2\log 3 + \frac{1}{2}\log^2 2 + \frac{1}{4}\log^2 3\right\}$$

$$+ \frac{3}{4}L'(1,\psi) - \frac{\log 2}{36}\sqrt{3}\pi.$$

Finally, we note that the numerical values of the explicit expressions in Examples 1–3 agree with the computed values in Table 1.

7. Connections with Generalized Gamma Functions

Let $\psi(z)$ be the logarithmic derivative $\psi(z) = \Gamma'(z)/\Gamma(z)$ of the gamma function. Lehmer [15, p. 133] proved the following relationship between $\psi(z)$ and $\gamma_0(r,m)$:

(7.1) $\gamma_0(r,m) = -\frac{1}{m}\left\{\psi\left(\frac{r}{m}\right) + \log m\right\}.$

In this section we shall prove an analogue of (7.1) involving $\gamma_k(r,m)$ for arbitrary $k \geq 0$. For this purpose, we introduce higher analogues of the ψ-function as follows. For integers $k \geq 0$ and for complex $z \neq 0, -1, -2, \ldots$ let

(7.2) $\psi_k(z) := -\gamma_k - \frac{\log^k z}{z} - \sum_{\nu=1}^{\infty}\left\{\frac{\log^k(\nu+z)}{\nu+z} - \frac{\log^k \nu}{\nu}\right\}.$

Note that for $k = 0$ this is a well-known representation for $\psi(z)$. For general k, this function occurs in Ramanujan's second notebook (see [3, Chapter 8,

Entry 22]). The case $k = 1$ was studied by Deninger [8]. The function $\psi_k(z)$ can be considered as the logarithmic derivative of an analogue $\Gamma_k(z)$ of the gamma function; for properties of this $\Gamma_k(z)$ and further connections with the γ_k, see [9].

Proposition 11. *For $k \geq 0$, $m \geq 1$, and $1 \leq r \leq m$ we have*

$$(7.3) \qquad \gamma_k(r, m) = \frac{-1}{m} \left\{ \frac{\log^{k+1} m}{k+1} + \sum_{\nu=0}^{k} \binom{k}{\nu} (\log m)^{k-\nu} \psi_\nu \left(\frac{r}{m} \right) \right\}.$$

In particular, for $k = 0$ we get (7.1), and $k = 1$ gives

$$(7.4) \qquad \gamma_1(r, m) = \frac{-1}{m} \left\{ \frac{1}{2} \log^2 m + (\log m) \psi \left(\frac{r}{m} \right) + \psi_1 \left(\frac{r}{m} \right) \right\}.$$

Proof of Proposition 11. We begin with a slight modification of the definition (1.4) and (1.3), valid for $1 \leq r \leq m$,

$$(7.5) \qquad \gamma_k(r, m) = \lim_{n \to \infty} \left\{ \sum_{j=0}^{n} \frac{\log^k(jm+r)}{jm+r} - \frac{\log^{k+1}(nm+r)}{m(k+1)} \right\}.$$

We use binomial expansions to obtain

$$\log^k(jm+r) = \sum_{\nu=0}^{k} \binom{k}{\nu} \log^{k-\nu} m \log^\nu \left(j + \frac{r}{m} \right)$$

and

$$\frac{\log^{k+1}(nm+r)}{k+1} = \sum_{\nu=0}^{k+1} \frac{1}{k+1} \binom{k+1}{\nu} \log^{k+1-\nu} m \log^\nu \left(n + \frac{r}{m} \right)$$

$$= \frac{\log^{k+1} m}{k+1} + \sum_{\nu=0}^{k} \binom{k}{\nu} \log^{k-\nu} m \frac{\log^{\nu+1}(n + r/m)}{\nu+1}.$$

Hence,

$$(7.6) \qquad \begin{aligned} \gamma_k(r, m) = {} & \frac{-\log^{k+1} m}{m(k+1)} + \frac{1}{m} \sum_{\nu=0}^{k} \binom{k}{\nu} (\log m)^{k-\nu} \\ & \times \lim_{n \to \infty} \left\{ \sum_{j=0}^{m} \frac{\log^\nu(j + r/m)}{j + r/m} - \frac{\log^{\nu+1}(n + r/m)}{\nu+1} \right\}. \end{aligned}$$

On the other hand, we have

$$(7.7) \qquad \gamma_\nu = \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{\log^\nu j}{j} - \frac{\log^{\nu+1} n}{\nu+1} \right\}.$$

We obtain now (7.3) by subtracting (7.7) from (7.6), using (7.2) and the fact that for fixed z,

$$\lim_{n \to \infty} \{\log^{\nu+1}(n+z) - \log^{\nu+1} n\} = 0.$$

This last limit can be found by way of a binomial expansion of $\log^{\nu+1}(n+z) = (\log n + \log(1 + z/n))^{\nu+1}$. The proof is now complete. $\square$

Remark. From (7.2) it follows that $\psi_k(1) = -\gamma_k$. Thus, (7.3) with $r = m$ implies (2.4)

We give now two applications of Proposition 11. The first one is a special case of a result of Ramanujan (see [3, p. 222]).

Corollary 11. *For $k \geq 0$, $m \geq 1$, and $1 \leq r \leq m$ we have*

$$
\frac{1}{m} \sum_{r=1}^{m-1} \psi_k\left(\frac{r}{m}\right) = (-1)^{k+1} \frac{\log^{k+1} m}{k+1} - \left(1 - \frac{1}{m}\right)\gamma_k
$$
(7.8)
$$
- \sum_{j=1}^{k} \binom{k}{j} (-1)^j (\log m)^j \gamma_{k-j}
$$

and in particular

$$
(7.9) \qquad \sum_{r=1}^{m-1} \psi\left(\frac{r}{m}\right) = -m \log m - (m-1)\gamma,
$$

$$
(7.10) \qquad \sum_{r=1}^{m-1} \psi_1\left(\frac{r}{m}\right) = m \frac{\log^2 m}{2} - (m-1)\gamma_1 + m(\log m)\gamma.
$$

Proof. We sum both sides of (7.3) over all r, $1 \leq r \leq m-1$, and use (2.8) and (2.4). Then

$$
\gamma_k = -\frac{\log^{k+1} m}{k+1} - \frac{1}{m} \sum_{\nu=0}^{k} \binom{k}{\nu} (\log m)^{k-\nu} \left(\sum_{r=1}^{m-1} \psi_\nu\left(\frac{r}{m}\right) - \gamma_\nu \right).
$$

To simplify notation, we set

$$
A_\nu := \sum_{r=1}^{m-1} \psi_\nu\left(\frac{r}{m}\right) - \gamma_\nu;
$$

then

$$
(7.11) \qquad A_k = -m\gamma_k - m\frac{\log^{k+1} m}{k+1} - \sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log m)^{k-\nu} A_\nu.
$$

We prove now (7.8) by induction. For $k = 0$, the relation (7.11) gives (7.9) immediately. Now suppose (7.8) is true up to $k-1$. We substitute A_ν from (7.8) into (7.11) and get

$$
A_k = -m\gamma_k - m\frac{\log^{k+1} m}{k+1} + mS_k
$$
(7.12)
$$
+ m \sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log m)^{k-\nu}\gamma_\nu + mT_k,
$$

where

$$
S_k := \sum_{\nu=0}^{k-1} \binom{k}{\nu} \frac{(-1)^\nu}{\nu+1} (\log m)^{k-\nu} (\log m)^{\nu+1}
$$

$$
= \frac{\log^{k+1} m}{k+1} \sum_{\nu=0}^{k-1} \binom{k+1}{\nu+1} (-1)^\nu
$$

and therefore

$$(7.13) \qquad S_k = (1 - (-1)^k)\frac{\log^{k+1} m}{k+1},$$

and where

$$T_k := \sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log m)^{k-\nu} \sum_{j=1}^{\nu} \binom{\nu}{j} (-1)^j (\log m)^j \gamma_{\nu-j}$$

$$= \sum_{\nu=0}^{k-1} \sum_{j=0}^{\nu-1} \binom{k}{\nu} \binom{\nu}{j} (-1)^{\nu-j} (\log m)^{k-j} \gamma_j$$

$$= \sum_{j=0}^{k-2} \binom{k}{j} (\log m)^{k-j} \gamma_j \sum_{\nu=j+1}^{k-1} \binom{k-j}{\nu-j} (-1)^{\nu-j}.$$

The inner sum in this last expression is

$$\sum_{\nu=1}^{k-j-1} \binom{k-j}{\nu} (-1)^\nu = -1 - (-1)^{k-j},$$

so that

$$T_k = -\sum_{j=0}^{k-2} \binom{k}{j} (\log m)^{k-j} \gamma_j - \sum_{j=0}^{k-2} \binom{k}{j} (-1)^{k-j} (\log m)^{k-j} \gamma_j$$

$$= -\sum_{j=0}^{k-1} \binom{k}{j} (\log m)^{k-j} \gamma_j - \sum_{j=1}^{k} \binom{k}{j} (-1)^j (\log m)^j \gamma_{k-j}.$$

This, with (7.13) and (7.12), proves (7.8). $\square$

Remark. If we set $m = 2$ in (7.8), we get

$$(7.14) \qquad \psi_k\left(\frac{1}{2}\right) = (-1)^{k+1} 2\frac{\log^{k+1} 2}{k+1} - \gamma_k - 2\sum_{j=1}^{k} \binom{k}{j} (-1)^j (\log 2)^j \gamma_{k-j}.$$

Corollary 12. *For $j \geq 0$ and $m \geq 1$, denote*

$$M_j(m) := \sum_{d \mid m} \mu\left(\frac{m}{d}\right) d \log^j d.$$

Then for all $k \geq 0$ and $m \geq 1$ we have

$$(7.15) \qquad \sum_{\substack{r=1 \\ (r,m)=1}}^{m} \psi_k\left(\frac{r}{m}\right) = \frac{(-1)^{k+1}}{k+1} M_{k+1}(m) - \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_j(m) \gamma_{k-j}.$$

Proof. We sum (7.3) over all r, $1 \leq r \leq m$, with $(r, m) = 1$, multiply both sides by m, and equate the left-hand side with the right-hand side of (4.5). If we denote

$$b_\nu := \sum_{\substack{r=1 \\ (r,m)=1}}^{m} \psi_\nu\left(\frac{r}{m}\right),$$

we get

$$b_k = \frac{1}{k+1} \left\{ N_{k+1}(m) - \varphi(m) \log^{k+1} m \right\}$$
$$- \sum_{\nu=0}^{k-1} \binom{k}{\nu} (\log m)^{k-\nu} b_\nu - \sum_{\nu=0}^{k} \binom{k}{\nu} N_{k-\nu}(m) \gamma_\nu.$$

By an induction similar to the one in the proof of Corollary 11, we find

$$b_k = \frac{1}{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j (\log m)^j N_{k+1-j}(m)$$
$$- \sum_{j=0}^{k} \binom{k}{j} (-1)^j \gamma_{k-j} \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu (\log m)^{j-\nu} N_\nu(m).$$

Using (4.2) and changing the order of summation, we get

$$\sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu (\log m)^{j-\nu} N_\nu(m) = M_j(m).$$

The assertion (7.15) now follows immediately. □

Remarks. 1. With $k = 0$, and using the facts that $M_0(m) = N_0(m)$ and $M_1(m) = (\log m) N_0(m) - N_1(m)$, we get with (4.3) and (4.4),

$$\sum_{\substack{r=1 \\ (r,m)=1}}^{m-1} \psi(r, m) = -\varphi(m) \left\{ \gamma + \log m + \sum_{p \mid m} \frac{\log p}{p-1} \right\};$$

this was derived by Lehmer [15].

2. If m is prime, then we have $M_j(m) = m \log^j m$ for $j \geq 1$ and $M_0(m) = m - 1$, and (7.15) coincides with (7.8).

As a final remark in this section, we consider (7.4) and recall that the $\gamma_1(r, m)$ can be given explicitly in terms of γ, γ_1, and some other known constants for $m = 2, 3, 4$, and 6 (see §6). The same is true for the $\psi(r/m)$ (see, e.g., [3, p. 184]). This allows us to give explicit expressions for $\psi_1(r/m)$ with $m = 2, 3, 4$, and 6. These can be considered as properties of the function $R(x)$ defined by Deninger [8].

Example. Let $r = 1$ and $m = 3$. It is known that

$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{3}{2} \log 3 - \frac{\pi}{2\sqrt{3}}.$$

Using (6.8) and (7.4), we get

$$\psi_1\left(\frac{1}{3}\right) = -\gamma_1 + \frac{1}{2}\left(3\log 3 + \frac{\pi}{\sqrt{3}}\right)\gamma + \frac{3}{4}\log^2 3$$
$$+ \pi\sqrt{3}\left\{\frac{2}{3}\log 2\pi - \frac{1}{12}\log 3 - \log\Gamma\left(\frac{1}{3}\right)\right\}.$$

Incidentally, with (7.2), we can thus evaluate the series

$$\sum_{\nu=1}^{\infty} \left\{ \frac{\log(\nu + 1/3)}{\nu + 1/3} - \frac{\log \nu}{\nu} \right\}.$$

Note that the constant γ_1 is cancelled.

Finally, we note that (7.3), considered as a linear system, can be solved for the $\psi_\nu(r/m)$ in terms of $\gamma_k(r, m)$ (see Corollary 3).

8. NUMERICAL EVALUATION OF THE $\gamma_k(r, m)$

Lehmer [15] computed the $\gamma_0(r, m)$ for $m \leq 9$ by use of the Euler-Maclaurin summation formula. This method can also be used to evaluate the $\gamma_k(r, m)$ for $k \geq 1$.

The generalized Euler constants γ_k, $1 \leq k \leq 19$, were evaluated by Liang and Todd [16] with the aid of the slightly modified Euler-Maclaurin formula

$$
(8.1) \quad \lim_{m \to \infty} \left\{ \sum_{j=1}^{m} f(j) - \int_1^m f(t)\,dt \right\} = \sum_{j=1}^{n} f(j) - \int_1^n f(t)\,dt - \frac{1}{2}f(n)
$$
$$
- \sum_{j=1}^{M} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(n) + R_M(n, \infty),
$$

with

$$(8.2) \qquad R_M(p, q) = \frac{1}{(2M+1)!} \int_p^q B_{2M+1}(t - [t]) f^{(2M+1)}(t)\,dt,$$

where $B_{2M+1}(x)$ is the Bernoulli polynomial of degree $2M + 1$ and B_{2j} is the $2j$th Bernoulli number (see, e.g., [1, p. 804]). For (8.1) to hold, we need that f have $2M + 1$ continuous derivatives, and $f^{(\nu)} \to 0$ as $t \to \infty$ for $\nu = 1, 2, \ldots, 2M + 1$. The parameters n and M are to be chosen such that the error $R_M(n, \infty)$ stays within prescribed bounds.

It is clear from the definition of the $\gamma_k(r, m)$ that for our purposes we have to take

$$(8.3) \qquad f(t) = \frac{\log^k(mt + r)}{mt + r}, \qquad m \geq 1, \ 1 \leq r \leq m.$$

The derivatives in (8.1) and (8.2) were computed in [16] by way of a multiple recursion, which would be fairly easy to program. However, we shall use an expression involving Stirling numbers of the first kind $s(n, k)$ (see, e.g., [7, p. 212 ff.] or [1, p. 824]).

Lemma 2. Let $f_k(x) := \log^k x / x$. Then

$$(8.4) \qquad f_k^{(\nu)}(x) = \frac{k!}{x^{\nu+1}} \sum_{i=0}^{k} s(\nu + 1, k + 1 - i) \frac{\log^i x}{i!}.$$

Proof. We use the following properties of the Stirling numbers of the first kind:

$$(8.5) \qquad s(\nu + 1, \mu) = 0 \quad \text{if } \mu = 0 \text{ or } \mu > \nu + 1,$$

$$(8.6) \qquad s(\nu + 1, \nu + 1) = 1,$$

$$(8.7) \qquad s(\nu, \mu) - \nu s(\nu, \mu + 1) = s(\nu + 1, \mu + 1) \quad \text{for } \nu, \mu \geq 0.$$

We prove now (8.4) by induction over ν. For $\nu = 0$, the relation (8.4), with (8.5) and (8.6), gives $f_k(x) = \log^k x / x$. Now suppose (8.4) is true for ν. Differentiating both sides, we obtain

$$
f_k^{(\nu+1)}(x) = \frac{k!}{x^{\nu+2}} \left\{ \sum_{i=1}^{k} s(\nu+1, k+1-i) \frac{\log^{i-1} x}{(i-1)!} \right.
$$

$$
\left. - (\nu+1) \sum_{i=0}^{k} s(\nu+1, k+1-i) \frac{\log^i x}{i!} \right\}
$$

$$
= \frac{k!}{x^{\nu+2}} \sum_{i=0}^{k} \{ s(\nu+1, k-i) - (\nu+1)s(\nu+1, k+1-i) \} \frac{\log^i x}{i!},
$$

where we have used (8.5). We get now (8.4) for $\nu+1$ by applying (8.7).   □

Remark. In the case $k = 1$, we get

$$
f_1^{(\nu)}(x) = \frac{(-1)^\nu \nu!}{x^{\nu+1}} \left[\log x - \left(1 + \frac{1}{2} + \cdots + \frac{1}{\nu} \right) \right]
$$

(see, e.g., [16, p. 171] or [3, p. 197]). This follows from (8.4) with (8.6) and

$$
s(\nu+1, 1) = (-1)^\nu \nu!, \qquad s(\nu+1, 2) = (-1)^{\nu+1} \nu! \left(1 + \frac{1}{2} + \cdots + \frac{1}{\nu} \right).
$$

We return now to (8.1). With $f(t)$ as in (8.3), we get

$$
\int_0^n f(t)\, dt = \frac{1}{m} \left\{ \frac{\log^{k+1}(mn+r)}{k+1} - \frac{\log^{k+1} r}{k+1} \right\},
$$

and the left-hand side of (8.1) becomes, with (1.4),

$$
\gamma_k(r, m) + \frac{1}{m} \frac{\log^{k+1} r}{k+1}.
$$

The second term is cancelled in (8.1); with Lemma 2 we therefore get the first part of the following asymptotic representation.

Proposition 12. *For $k \geq 0$, $m \geq 1$, and $1 \leq r \leq m$ we have*

$$
(8.8) \quad \gamma_k(r, m) = \sum_{j=0}^{n} \frac{\log^k(mj+r)}{mj+r} - \frac{1}{m} \frac{\log^{k+1}(mn+r)}{k+1} - \frac{1}{2} \frac{\log^k(mn+r)}{mn+r}
$$

$$
- \sum_{j=1}^{M} \frac{k!}{(2j)!} \frac{B_{2j}}{m(n+r/m)^{2j}} \sum_{i=0}^{k} s(2j, k+1-i) \frac{\log^i(mn+r)}{i!}
$$

$$
+ R_M(n, \infty).
$$

If $n > e^{k/7} - 1$, then

$$
(8.9) \qquad |R_M(n, \infty)| \leq \frac{2}{1 - 2^{-2M}} \frac{(2M+2)!}{(2\pi n)^{2M+1}} \frac{k! \log^k(mn+r)}{2M-6}.
$$

Proof. It remains to estimate the error term $R_M(n, \infty)$. With f as in (8.3), we have by (8.4)

$$f^{(2M+1)}(t) = \frac{k! m^{2M+1}}{(mt+r)^{2M+2}} \sum_{i=0}^{k} \frac{1}{i!} s(2M+2, k+1-i) \log^i(mt+r)$$

and therefore, using (8.5),

$$|f^{(2M+1)}(t)| \le k! m^{2M+1} \sum_{j=0}^{2M+2} |s(2M+2, j)| \frac{\log^k(mt+r)}{(mt+r)^{2M+2}}.$$

The sum of the "unsigned" Stirling numbers is $(2M+2)!$ (see, e.g., [7, p. 213]). Hence,

$$(8.10) \qquad |f^{(2M+1)}(t)| \le k! (2M+2)! m^{2M+1} \frac{\log^k(mt+r)}{(mt+r)^{2M+2}}.$$

Furthermore, we have

$$|B_{2M+1}(t-[t])| < \frac{1}{1-2^{-2M}} \frac{2(2M+1)!}{(2\pi)^{2M+1}}$$

(see, e.g., [1, p. 805]), so that, with (8.10) and (8.2), we get

$$(8.11) \qquad |R_M(n, \infty)| \le \frac{2k! m^{2M+1}}{1-2^{-2M}} \frac{(2M+2)!}{(2\pi)^{2M+1}} \int_n^\infty \frac{\log^k(mt+r)}{(mt+r)^{2M+2}} \, dt.$$

It is easy to verify that $\log^k(mt+r)/(mt+r)^7$ is decreasing as a function of t if $\log(mt+r) > k/7$. Since $t \ge n$ and $m \ge 1$, $r \ge 1$, this holds when

$$(8.12) \qquad n > e^{k/7} - 1.$$

Hence, under condition (8.12) we have

$$|R_M(n, \infty)| \le \frac{2k! m^{2M+1}}{1-2^{-2M}} \frac{(2M+2)!}{(2\pi)^{2M+1}} \frac{\log^k(mn+r)}{(mn+r)^7} \int_n^\infty (mt+r)^{-2M+5} \, dt.$$

Upon integrating and using the fact that $mn + r > mn$, we obtain the estimate (8.9). □

Equation (8.8) proves to be convenient for computation with the symbolic manipulation package MAPLE. Bernoulli and Stirling numbers are "known" to MAPLE; hence minimal programming effort is required. The $\gamma_k(r, m)$ were computed for $1 \le k \le 20$, $1 \le m \le 9$, and $1 \le r \le m$. For $1 \le k \le 10$ one obtains, by (8.9), $|R_M(n, \infty)| < 0.15 \cdot 10^{-15}$ for $n = 40$ and $M = 10$. For $11 \le k \le 20$, the choice was $n = 40$ and $M = 20$; in this case, $|R_M(n, \infty)| < 0.31 \cdot 10^{-14}$. Because of high cancellations (see the remarks on this problem in [16]), all calculations were carried out to 40 digits accuracy; Table 1 in the Supplement section at the end of this issue shows 12 significant digits. The values for $\gamma_k(1, 1)$, $1 \le k \le 20$, agree with those for γ_k in the tables in [16, 4].

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